WNE Linear Algebra
Final Exam
Series A

2 February 2023

Please use separate sheets for different problems. Please use single sheet for all questions. Give reasons to your answers. Please provide the following data on each sheet

- **name, surname and your student number,**
- **number of your group,**
- **number of the corresponding problem and the series.**

Each problem is worth 10 marks. Each question is worth 4 marks.

Problems

Problem 1.
Let $V = \text{lin}((1,1,4),(2,1,6),(1,2,6))$ be a subspace of $\mathbb{R}^3$.

a) find a basis $\mathcal{A}$ of the subspace V and the dimension of V,
b) for which $t \in \mathbb{R}$ does the vector $v = (-1,1,t)$ belong to V? for all such $t \in \mathbb{R}$ find coordinates of v relative to basis $\mathcal{A}$.

Problem 2.
Let $V \subset \mathbb{R}^4$ be a subspace given by the homogeneous system of linear equations

$$\begin{cases} x_1 + 2x_2 + 10x_3 + 2x_4 = 0 \\ 2x_1 + 3x_2 + 18x_3 + 5x_4 = 0 \\ x_1 + 6x_2 + 18x_3 - 2x_4 = 0 \end{cases}$$

a) find a basis and the dimension of the subspace V,
b) for which $t \in \mathbb{R}$ is the subspace V contained in the subspace W_t, i.e. $V \subset W_t$, where

$$W_t = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_1 + x_2 + 8x_3 + tx_4 = 0\}?$$

Problem 3.
Let

$$A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 3 & 4 & 3 & 2 \\ 3 & 5 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 4 & 7 \\ 0 & 0 & 3 & 5 \\ 4 & 5 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{bmatrix}.$$

a) compute $\det A$,
b) compute $\det(AB^{\mathsf{T}}AB)$.

Problem 4.
Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear endomorphism given by the formula

$$\varphi((x_1,x_2)) = (2x_2, -4x_1 + 6x_2).$$

a) find the eigenvalues of φ and bases of the corresponding eigenspaces. Find a basis $\mathcal{A}$ of $\mathbb{R}^2$ consisting of eigenvectors of φ.
b) find the matrix $M(\varphi \circ \varphi)_{st}^{\mathcal{A}}$.

Problem 5.
Let $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 - x_2 + x_3 = 0\}$ be a subspace of $\mathbb{R}^3$.
a) find an orthonormal basis of V,
b) compute the orthogonal projection of $w = (5, 1, 2)$ onto V.

Problem 6.
Consider the following linear programming problem $x_1 - x_3 \to \min$ in the standard form with constraints

$$\begin{cases} -x_1 + 2x_2 - x_3 + x_4 = 10 \\ 2x_1 + x_2 + 2x_3 + 3x_4 = 15 \end{cases} \quad \text{and } x_i \geqslant 0 \text{ for } i = 1, \ldots, 4.$$

a) which of the sets $\mathcal{B}_1 = \{1,3\}$, $\mathcal{B}_2 = \{1,2\}$, $\mathcal{B}_3 = \{1,4\}$ is basic feasible? Write the corresponding basic solution for all basic sets,
b) solve the linear programming problem using simplex method.

Questions

Question 1.
Let $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^2$ be an endomorphism given by the formula

$$\varphi((x_1, x_2)) = (4x_1 + tx_2, x_1 + 2x_2).$$

For which $t \in \mathbb{R}$ is vector $v = (1, 1)$ an eigenvector of φ? Find the corresponding eigenvalue.

Question 2.
Let $A \in M(n \times n; \mathbb{R})$ be a diagonalizable matrix. If $C^{-1}AC = D$ is a diagonal matrix for some invertible matrix $C \in M(n \times n; \mathbb{R})$, does it follow that columns of $(C^\intercal)^{-1}$ are eigenvectors of matrix $A^\intercal$?

Question 3.
If $A \in M(2 \times 2; \mathbb{R})$ is an antisymmetric matrix, i.e. $A^\intercal = -A$, and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the unit matrix, does it follow that matrix $A - I$ is invertible?

Question 4.
Matrix $M(P_V)_{st}^{st} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$ is a matrix of an orthogonal projection P_V onto some subspace $V \subset \mathbb{R}^2$. Find an orthonormal basis of $V^\perp$.

Question 5.
Vectors $(1, 1)$ and $(1, 3)$ are (some) solutions of a system of linear equations in two variables. Given that $(0, 0)$ is not a solution, find an example of a third solution of that system different from the two others.